

SHOKUROV'S ACC CONJECTURE FOR LOG CANONICAL THRESHOLDS ON SMOOTH VARIETIES

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ABSTRACT. Shokurov conjectured that the set of all log canonical thresholds on varieties of bounded dimension satisfies the ascending chain condition. In this paper we prove that the conjecture holds for log canonical thresholds on smooth varieties and, more generally, on locally complete intersection varieties and on varieties with quotient singularities.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero. Log canonical varieties are varieties with mild singularities that provide the most general context for the Minimal Model Program. More generally, one considers the log canonicity condition on pairs (X, \mathfrak{a}^t) , where \mathfrak{a} is a proper ideal sheaf on X (most of the times, it is the ideal of an effective Cartier divisor), and t is a nonnegative real number. Given a log canonical variety X over k , and a proper nonzero ideal sheaf \mathfrak{a} on X , one defines the *log canonical threshold* $\text{lct}(\mathfrak{a})$ of the pair (X, \mathfrak{a}) . This is the largest number t such that the pair (X, \mathfrak{a}^t) is log canonical. One makes the convention $\text{lct}(0) = 0$ and $\text{lct}(\mathcal{O}_X) = \infty$. The log canonical threshold is a fundamental invariant in birational geometry, see for example [Kol1], [EM2], or Chapter 9 in [Laz].

Shokurov's ACC Conjecture [Sho] says that the set of all log canonical thresholds on varieties of any fixed dimension satisfies the ascending chain condition, that is, it contains no infinite strictly increasing sequences. This conjecture attracted considerable interest due to its implications to the Termination of Flips Conjecture (see [Bir] for a result in this direction). The first unconditional results on sequences of log canonical thresholds on smooth varieties of arbitrary dimension have been obtained in [dFM], and they were subsequently reproved and strengthened in [Kol2].

Key words and phrases. Log canonical threshold, ascending chain condition, inversion of adjunction, \mathfrak{m} -adic approximation, connectedness theorem.

2000 Mathematics Subject Classification. Primary 14E15; Secondary 14B05, 14E30.

The first author was partially supported by NSF CAREER grant DMS-0847059, the second author was partially supported by NSF grant DMS-0700774, and the third author was partially supported by NSF grant DMS-0758454, and by a Packard Fellowship.

The main goal of this paper is to prove Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties and, more generally, on varieties that are locally complete intersection (l.c.i. for short). Our first result deals with the smooth case.

Theorem 1.1. *For every n , the set*

$$\mathcal{T}_n^{\text{sm}} := \{\text{lct}(\mathfrak{a}) \mid X \text{ is smooth, } \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

of log canonical thresholds on smooth varieties of dimension n satisfies the ascending chain condition.

As we will see, every log canonical threshold on a variety with quotient singularities can be written as a log canonical threshold on a smooth variety of the same dimension. Therefore for every n the set

$$\mathcal{T}_n^{\text{quot}} := \{\text{lct}(\mathfrak{a}) \mid X \text{ has quotient singularities, } \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

is equal to $\mathcal{T}_n^{\text{sm}}$, and thus the ascending chain condition also holds for log canonical thresholds on varieties with quotient singularities.

In order to extend the result to log canonical thresholds on locally complete intersection varieties, we consider a more general version of log canonical thresholds. Given a variety X and an ideal sheaf \mathfrak{b} on X such that the pair (X, \mathfrak{b}) is log canonical, for every nonzero ideal sheaf $\mathfrak{a} \subsetneq \mathcal{O}_X$ we define the *mixed log canonical threshold* $\text{lct}_{(X, \mathfrak{b})}(\mathfrak{a})$ to be the largest number c such that the pair $(X, \mathfrak{b} \cdot \mathfrak{a}^c)$ is log canonical. Note that when $\mathfrak{b} = \mathcal{O}_X$, this is nothing but $\text{lct}(\mathfrak{a})$. Again, one sets $\text{lct}_{(X, \mathfrak{b})}(0) = 0$ and $\text{lct}_{(X, \mathfrak{b})}(\mathcal{O}_X) = \infty$. The following is our main result.

Theorem 1.2. *For every n , the set*

$$\mathcal{M}_n^{\text{l.c.i.}} := \{\text{lct}_{(X, \mathfrak{b})}(\mathfrak{a}) \mid X \text{ is l.c.i., } \dim X = n, \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X, \mathfrak{a} \neq \mathcal{O}_X, (X, \mathfrak{b}) \text{ log canonical}\}$$

of mixed log canonical thresholds on l.c.i. varieties of dimension n satisfies the ascending chain condition.

By restricting to the case $\mathfrak{b} = \mathcal{O}_X$, we obtain the following immediate corollary.

Corollary 1.3. *For every n , the set*

$$\mathcal{T}_n^{\text{l.c.i.}} := \{\text{lct}(\mathfrak{a}) \mid X \text{ is log canonical and l.c.i., } \dim X = n, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

of log canonical thresholds on log canonical l.c.i. varieties of dimension n satisfies the ascending chain condition.

We will use Inversion of Adjunction (in the form treated in [EM1]) to reduce Theorem 1.2 to the analogous statement in which X ranges over smooth varieties. More precisely, we show that all sets

$$\mathcal{M}_n^{\text{sm}} := \{\text{lct}_{(X, \mathfrak{b})}(\mathfrak{a}) \mid X \text{ is smooth, } \dim X = n, \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X, \mathfrak{a} \neq \mathcal{O}_X, (X, \mathfrak{b}) \text{ log canonical}\}$$

satisfy the ascending chain condition. It follows by Inversion of Adjunction that every mixed log canonical threshold of the form $\text{lct}_{(X,\mathfrak{b})}(\mathfrak{a})$, with \mathfrak{a} and \mathfrak{b} ideal sheaves on an l.c.i. variety X , can be expressed as a mixed log canonical threshold on a (typically higher dimensional) smooth variety. This is the step that requires us to work with mixed log canonical thresholds. The key observation that makes this approach work is that if X is an l.c.i. variety with log canonical singularities, then $\dim_k T_x X \leq 2 \dim X$ for every $x \in X$. This implies that the above reduction to the smooth case keeps the dimension of the ambient variety bounded.

The proofs of the above results use a general method of associating to a sequence of ideals of polynomials over a field k , an ideal of power series over a field extension of k . The original construction considered in [dFM] is a standard application of nonstandard methods, and relies on the use of ultrafilters. This construction was subsequently replaced in [Kol2] by a purely algebro-geometric construction, that gives a *generic limit* by using a sequence of \mathfrak{m} -adic approximations and field extensions. The two constructions are similar in nature, and either construction can be employed to prove the results of this paper. We chose to present the proofs using the second construction, which is geometrically more explicit.

A key ingredient is the following effective \mathfrak{m} -adic semicontinuity property for log canonical thresholds (that we will only use in the case when $X = \mathbf{A}^n$ and E lies over a point of \mathbf{A}^n).

Theorem 1.4. *Let X be a log canonical variety, and let $\mathfrak{a} \subsetneq \mathcal{O}_X$ be a proper ideal. Suppose that E is a prime divisor over X computing $\text{lct}(\mathfrak{a})$, and consider the ideal sheaf $\mathfrak{q} := \{h \in \mathcal{O}_X \mid \text{ord}_E(h) > \text{ord}_E(\mathfrak{a})\}$. If $\mathfrak{b} \subseteq \mathcal{O}_X$ is an ideal such that $\mathfrak{b} + \mathfrak{q} = \mathfrak{a} + \mathfrak{q}$, then after possibly restricting to an open neighborhood of the center of E , we have $\text{lct}(\mathfrak{b}) = \text{lct}(\mathfrak{a})$.*

This result (for principal ideals) was first proven by Kollár in [Kol2] using deep results in the Minimal Model Program from [BCHM] and a theorem on Inversion of Adjunction from [Kaw]. We give an elementary proof of Theorem 1.4 which only uses the Connectedness Theorem of Shokurov and Kollár (see Theorem 7.4 in [Kol1]). We note that in the case of a divisor E with zero-dimensional center, Kollár's proof extends to cover also ideals in a power series ring, and this fact is important for his approach. In fact, as we will see, this version can be formally deduced from the statement of Theorem 1.4 (see Corollary 3.5).

It is interesting to observe how, in the end, all the results of this paper only rely on basic facts in birational geometry, such as Resolution of Singularities and the Connectedness Theorem and, for the l.c.i. case, on Inversion of Adjunction. We expect however that new ideas and more sophisticated techniques will be necessary to tackle the ACC Conjecture in its general formulation.

Acknowledgment. We are grateful to Shihoko Ishii and Angelo Vistoli for useful discussions and correspondence, and to János Kollár for his comments and suggestions on

previous versions of our work. Furthermore, as we have already mentioned, two key ideas we use in this paper come from Kollár's article [Kol2].

2. GENERALITIES ON LOG CANONICAL THRESHOLDS

Let k be a field of characteristic zero. In what follows X will be either a normal and \mathbf{Q} -Gorenstein variety over k , or $\text{Spec}(k[[x_1, \dots, x_n]])$.

We recall the definition of log canonical threshold in a slightly more general version, and discuss some of the properties that will be needed later. For the basic facts about log canonical pairs in the setting of algebraic varieties, see [Kol1] or Chapter 9 in [Laz], while for the case of the spectrum of a formal power series ring we refer to [dFM]. The key point is that by [Tem], log resolutions exist also in the latter case, and therefore the usual theory of log canonical pairs carries through.

Suppose that X is as above. Let \mathfrak{a} and \mathfrak{b} be nonzero coherent sheaves of ideals on X with $\mathfrak{a} \neq \mathcal{O}_X$, and assume that the pair (X, \mathfrak{b}) is log canonical. We consider the following relative version of the definition of log canonical threshold (there is an analogous definition in the language of \mathbf{Q} -divisors that is broadly used in the literature): we define the *mixed log canonical threshold* of \mathfrak{a} with respect to the pair (X, \mathfrak{b}) to be

$$\text{lct}_{(X, \mathfrak{b})}(\mathfrak{a}) := \sup\{c \geq 0 \mid (X, \mathfrak{b} \cdot \mathfrak{a}^c) \text{ is log canonical}\}.$$

Whenever the ambient variety X is understood, we drop it from the notation, and simply write $\text{lct}_{\mathfrak{b}}(\mathfrak{a})$. Observe that in the case $\mathfrak{b} = \mathcal{O}_X$, the mixed log canonical threshold $\text{lct}_{\mathcal{O}_X}(\mathfrak{a})$ is nothing else than the usual *log canonical threshold* $\text{lct}(\mathfrak{a})$ of \mathfrak{a} . We make the convention $\text{lct}_{\mathfrak{b}}(0) = 0$ and $\text{lct}_{\mathfrak{b}}(\mathcal{O}_X) = \infty$.

The fact that log canonicity can be checked on a log resolution allows us to describe the mixed log canonical threshold in terms of any such resolution. Suppose that $\pi: Y \rightarrow X$ is a log resolution of $\mathfrak{a} \cdot \mathfrak{b}$, and write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}(-\sum_i a_i E_i)$, $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}(-\sum_i b_i E_i)$, and $K_{Y/X} = \sum_i k_i E_i$. Still assuming that \mathfrak{a} and \mathfrak{b} are nonzero ideals, $\mathfrak{a} \neq \mathcal{O}_X$, and (X, \mathfrak{b}) is log canonical (that is, $\text{lct}(\mathfrak{b}) \geq 1$), it follows from the characterization of log canonicity in terms of a log resolution that

$$(1) \quad \text{lct}_{\mathfrak{b}}(\mathfrak{a}) = \min \left\{ \frac{k_i + 1 - b_i}{a_i} \mid a_i > 0 \right\}.$$

We see from the above formula that the mixed log canonical threshold is a rational number. Note also that it is zero if and only if there is i such that $a_i > 0$ and $b_i = k_i + 1$ (in other words, if (X, \mathfrak{b}) is not Kawamata log terminal and there is a non-klt center contained in the zero-locus of \mathfrak{a}).

It is convenient to use also a local version of the (mixed) log canonical threshold. For every point $p \in V(\mathfrak{a})$ such that the pair (X, \mathfrak{b}) is log canonical in some neighborhood of p , if in (1) we take the minimum only over those i such that $p \in \pi(E_i)$, we get the *mixed log*

canonical threshold at p , denoted $\text{lct}_{(X, \mathfrak{b}), p}(\mathfrak{a})$. This is the maximum of $\text{lct}_{\mathfrak{b}|_U}(\mathfrak{a}|_U)$, when U ranges over the open neighborhoods of p . When $\mathfrak{b} = \mathcal{O}_X$, we simply write $\text{lct}_p(\mathfrak{a})$.

Remark 2.1. It follows from the description in terms of a log resolution that if $X = U_1 \cup \dots \cup U_r$, with U_j open, then $\text{lct}_{\mathfrak{b}}(\mathfrak{a}) = \min_j \text{lct}_{\mathfrak{b}|_{U_j}}(\mathfrak{a}|_{U_j})$.

Remark 2.2. If \mathfrak{b} and \mathfrak{a} are as above and $c := \text{lct}_{\mathfrak{b}}(\mathfrak{a})$, then $\text{lct}(\mathfrak{b} \cdot \mathfrak{a}^c) = 1$ (where, of course, $\text{lct}(\mathfrak{b} \cdot \mathfrak{a}^c)$ is the largest nonnegative q such that the pair $(X, \mathfrak{b}^q \cdot \mathfrak{a}^{qc})$ is log canonical). Indeed, by assumption the pair $(X, \mathfrak{b} \cdot \mathfrak{a}^c)$ is log canonical, and for every $\alpha > 1$ the pair $(X, (\mathfrak{b} \cdot \mathfrak{a}^c)^\alpha)$ is not log canonical since $(X, \mathfrak{b} \cdot \mathfrak{a}^{c\alpha})$ is not. Note however that the converse of this property does not hold: in fact, if $\text{lct}(\mathfrak{b}) = 1$ and the zero-locus of \mathfrak{a} does not contain any non-klt center of (X, \mathfrak{b}) , then $c = \text{lct}_{\mathfrak{b}}(\mathfrak{a}) > 0$ and $\text{lct}(\mathfrak{b} \cdot \mathfrak{a}^t) = 1$ for every $0 < t \leq c$.

Remark 2.3. Suppose that X , \mathfrak{a} and \mathfrak{b} are as above, with X smooth. For every $p \in V(\mathfrak{a})$, we have $\text{lct}_{(X, \mathfrak{b}), p}(\mathfrak{a}) = \text{lct}_{(X', \mathfrak{b}')}(\mathfrak{a}')$, where $X' = \text{Spec}(\widehat{\mathcal{O}_{X, p}})$, and \mathfrak{a}' , \mathfrak{b}' are the pull-backs of the ideals \mathfrak{a} and, respectively, \mathfrak{b} to X' . The argument follows as in the case $\mathfrak{b} = \mathcal{O}_X$, for which we refer to [dFM, Proposition 2.9].

We will adopt the following terminology.

Definition 2.4. Let X and $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ be as above. We say that a prime divisor E over X *computes* $\text{lct}_{\mathfrak{b}}(\mathfrak{a})$ if there is a log resolution $\pi: Y \rightarrow X$ such that, with the above notation, E induces the same valuation as a divisor E_i on Y for which $a_i > 0$ and the minimum in (1) is achieved for this i .

Suppose now that k is algebraically closed. For every $n \geq 0$, we consider the sets $\mathcal{T}_n^{\text{sm}}$, $\mathcal{T}_n^{\text{quot}}$, $\mathcal{T}_n^{\text{l.c.i.}}$, $\mathcal{M}_n^{\text{sm}}$ and $\mathcal{M}_n^{\text{l.c.i.}}$ defined in the Introduction. Note that for $n = 0$ all these sets are equal to $\{0\}$. It is convenient to extend the definition to $n < 0$ by declaring all these sets to be empty in this range. We will use the basic fact (cf. [dFM, Proposition 3.3]) that for every $n \geq 1$,

$$\mathcal{T}_n^{\text{sm}} = \{\text{lct}_0(\mathfrak{a}) \mid \mathfrak{a} \subseteq (x_1, \dots, x_n) \subset k[x_1, \dots, x_n]\}.$$

Similarly, for every $n \geq 1$ we have

$$\mathcal{M}_n^{\text{sm}} = \{\text{lct}_{(\mathbf{A}^n, \mathfrak{b}), 0}(\mathfrak{a}) \mid \mathfrak{a}, \mathfrak{b} \subseteq k[x_1, \dots, x_n], \mathfrak{a} \subseteq (x_1, \dots, x_n), \text{lct}_0(\mathfrak{b}) \geq 1\}.$$

The proof is analogous to the non-mixed case, and is left to the reader.

3. EFFECTIVE \mathfrak{m} -ADIC SEMICONTINUITY OF LOG CANONICAL THRESHOLDS

Let X be a log canonical variety defined over an algebraically closed field of characteristic zero k . We start by proving Theorem 1.4 in the special case of principal ideals.

Theorem 3.1. *Let E be a divisor over X , computing $\text{lct}(f)$ for some $f \in \mathcal{O}(X)$. If $g \in \mathcal{O}(X)$ is such that $\text{ord}_E(f - g) > \text{ord}_E(f)$, then after possibly replacing X by an open neighborhood of the center of E , we have $\text{lct}(f) = \text{lct}(g)$.*

The interesting inequality is $\text{lct}(g) \geq \text{lct}(f)$, the reverse one being trivial. Note that if the center of E on X is equal to a point $p \in X$, then whenever $\text{mult}_p(f - g) > \text{ord}_E(f)$, we have $\text{ord}_E(f - g) > \text{ord}_E(f)$, and the theorem gives $\text{lct}_p(g) = \text{lct}_p(f)$.

As already explained in the Introduction, a proof of the theorem was given in [Kol2] relying on deep results in the Minimal Model Program and on Inversion of Adjunction. We give an elementary proof, only using the Connectedness Theorem.

Proof of Theorem 3.1. The inequality $\text{lct}(f) \geq \text{lct}(g)$ is easy. Indeed, since $\text{ord}_E(f - g) > \text{ord}_E(f)$, we have $\text{ord}_E(g) = \text{ord}_E(f)$, and therefore, if Y is the model over X on which E lies, then

$$\text{lct}(g) \leq \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(g)} = \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(f)} = \text{lct}(f).$$

The first step in the proof of the reverse inequality is to reduce to the case when $\text{ord}_F(f - g) > \text{ord}_F(f)$ for *all* divisors F that compute $\text{lct}(f)$ on some log resolution of fg . In order to do this, let us choose a log resolution $\pi: Y \rightarrow X$ of $fg(f - g)$ such that the divisor E appears on Y . Let E_1, \dots, E_t be the irreducible components of the divisor $K_{Y/X} + \pi^*(\text{div}(fg(f - g)))$. After relabelling the indices, we may assume that $E = E_1$. In the following, we denote

$$a_i := \text{ord}_{E_i}(f), \quad b_i := \text{ord}_{E_i}(g), \quad \text{and} \quad k_i := \text{ord}_{E_i}(K_{Y/X}).$$

In order to prove the theorem, it is enough to show that for every $q \in \pi(E)$ we have $\text{lct}_q(g) \geq \text{lct}_q(f)$ (note that $\text{lct}_q(f) = \text{lct}(f)$). Fix such q . After possibly replacing X by an open neighborhood of q , we may assume that $q \in \pi(E_i)$ for every i .

For every $m \geq 1$, we consider $f_m := f^m h$ and $g_m := g^m h$, where $h = f - g$. Note that by assumption π is a log resolution for both f_m and g_m .

Lemma 3.2. *If $m \gg 1$, then*

i) E_i computes $\text{lct}(f_m)$ if and only if it computes $\text{lct}(f)$ and, in addition,

$$\frac{\text{ord}_{E_i}(f)}{\text{ord}_{E_i}(h)} = \min \left\{ \frac{\text{ord}_{E_j}(f)}{\text{ord}_{E_j}(h)} \mid E_j \text{ computes } \text{lct}(f) \right\}.$$

ii) For every i such that E_i computes $\text{lct}(f_m)$, we have $\text{ord}_{E_i}(f_m - g_m) > \text{ord}_{E_i}(f_m)$.

Proof. We put $c_i = \text{ord}_{E_i}(h)$. Since $m \gg 1$, we have

$$\frac{k_i + 1}{a_i + \frac{c_i}{m}} \leq \frac{k_j + 1}{a_j + \frac{c_j}{m}}$$

if and only if $\frac{k_i + 1}{a_i} \leq \frac{k_j + 1}{a_j}$, and either this inequality is strict, or $\frac{k_i + 1}{c_i} \leq \frac{k_j + 1}{c_j}$. This shows that every divisor E_i that computes $\text{lct}(f_m)$ also computes $\text{lct}(f)$. Furthermore, if E_i computes $\text{lct}(f)$, then it computes $\text{lct}(f_m)$ if and only if $\frac{k_i + 1}{c_i} \leq \frac{k_j + 1}{c_j}$ for every j such that

E_j computes $\text{lct}(f)$. Note that this holds if and only if $\frac{a_i}{c_i} \leq \frac{a_j}{c_j}$ (since $k_i + 1 = \text{lct}(f)a_i$ and $k_j + 1 = \text{lct}(f)a_j$), hence i).

Suppose now that E_i computes $\text{lct}(f_m)$. It follows from i) and our hypothesis that $\frac{a_i}{c_i} \leq \frac{a_1}{c_1} < 1$. Since $f_m - g_m = (f^m - g^m)h$, in order to prove ii) it is enough to show that $\text{ord}_{E_i}(f^m - g^m) > m \cdot \text{ord}_{E_i}(f)$. Note that $a_i < c_i$ implies $\text{ord}_{E_i}(f) = \text{ord}_{E_i}(g)$ (recall that $g = f - h$). We write

$$f^m - g^m = (g + h)^m - g^m = \sum_{\ell=1}^m \binom{m}{\ell} h^\ell g^{m-\ell}.$$

For every $\ell \geq 1$ we have $\text{ord}_{E_i}(h^\ell g^{m-\ell}) > m \cdot \text{ord}_{E_i}(f)$, hence $\text{ord}_{E_i}(f^m - g^m) > m \cdot \text{ord}_{E_i}(f)$. This completes the proof of the lemma. \square

Observe that $\text{lct}(f) = \lim_{m \rightarrow \infty} m \cdot \text{lct}(f_m)$ and $\text{lct}(g) = \lim_{m \rightarrow \infty} m \cdot \text{lct}(g_m)$. Indeed, it follows from definition that

$$\text{lct}(f_m) = \min_i \frac{k_i + 1}{ma_i + c_i} = \frac{1}{m} \cdot \min_i \frac{k_i + 1}{a_i + \frac{c_i}{m}},$$

which gives the first equality, and the second one follows in the same way. Thus, if we can prove the theorem for f_m and g_m in place of f and g , for all $m \gg 1$, then we deduce the statement for f and g .

Therefore, by Lemma 3.2, we are reduced to proving Theorem 3.1 in the case when there is a log resolution $\pi: Y \rightarrow X$ for fg such that for all divisors E_i on π that compute $\text{lct}(f)$ we have $\text{ord}_{E_i}(f - g) > \text{ord}_{E_i}(f)$. We shall thus assume that this is the case. We keep the notation previously introduced, so that in particular $a_i = \text{ord}_{E_i}(f)$ and $b_i = \text{ord}_{E_i}(g)$ for every i . Recall also that we may assume $q \in \pi(E_i)$ for all i .

Lemma 3.3. *Under the above assumptions, if E_i is a divisor computing $\text{lct}(f)$, then $\text{ord}_{E_j}(f) = \text{ord}_{E_j}(g)$ for every j such that $E_i \cap E_j \neq \emptyset$.*

Proof. Let $p \in E_i \cap E_j$ be a general point, and let $y_i, y_j \in \mathcal{O}_{Y,p}$ be part of a regular system of parameters, and generating the images in $\mathcal{O}_{Y,p}$ of the ideals defining E_i and E_j , respectively. We have in $\mathcal{O}_{Y,p}$

$$\pi^*(f) = uy_i^{a_i}y_j^{a_j} \quad \text{and} \quad \pi^*(g) = vy_i^{b_i}y_j^{b_j},$$

where $u, v \in \mathcal{O}_{Y,p}$ are invertible elements. By assumption, $\pi^*(f - g) = y_i^{a_i+1}w$ for some $w \in \mathcal{O}_{Y,p}$. This has two consequences. The first is that $b_i = a_i$. Furthermore, we see that $y_i^{-a_i}\pi^*(f)$ and $y_i^{-a_i}\pi^*(g)$ have the same restriction to E_i . This implies that $b_j = a_j$, which is the assertion in the lemma. \square

We can now finish the proof of Theorem 3.1. Let $c = \text{lct}(f)$, and for every i let

$$\alpha_i := ca_i - k_i \quad \text{and} \quad \beta_i := cb_i - k_i.$$

Note that $\alpha_i \leq 1$ for every i , and equality holds precisely for those i such that E_i computes $\text{lct}(f)$. The above lemma says that for every i such that $\alpha_i = 1$, we have $\beta_i = 1$, and more generally $\alpha_j = \beta_j$ for every j such that $E_i \cap E_j \neq \emptyset$.

To finish, we apply the main ingredient of the proof, namely, the Connectedness Theorem of Shokurov and Kollár (see Theorem 7.4 in [Kol1]), which in our case says that the union $\cup_{\beta_j \geq 1} E_j$ is connected in the neighborhood of $\pi^{-1}(q)$. Since $q \in \pi(E_i)$ for every i , this implies that $\cup_{\beta_j \geq 1} E_j$ is connected.

Let us look at an arbitrary divisor E_i that computes $\text{lct}(f)$, so that $\alpha_i = 1$. We have seen that in this case $\beta_i = 1$. If E_j is any other divisor that meets E_i and such that $\beta_j \geq 1$, then we have $1 \geq \alpha_j = \beta_j \geq 1$ by Lemma 3.3, and therefore $\alpha_j = \beta_j = 1$. This implies by induction on s that for every sequence of divisors $E_i, E_{j_1}, \dots, E_{j_s}$ such that any two consecutive divisors intersect, and such that $\beta_{j_\ell} \geq 1$ for all ℓ , we have $\alpha_{j_\ell} = \beta_{j_\ell} = 1$ for every ℓ . Since the set $\cup_{\beta_j \geq 1} E_j$ is connected, we conclude that $\beta_j \leq 1$ for every j , and thus $\text{lct}(g) \geq c$. This completes the proof of Theorem 3.1. \square

Remark 3.4. The above proof also gives the following statement. Suppose that f and g are as in Theorem 3.1, such that for *all* divisors E_i over X computing $\text{lct}(f) = c$, we have $\text{ord}_{E_i}(f - g) > \text{ord}_{E_i}(f)$ (it is easy to see that it is enough to check this condition only on the divisors on a fixed log resolution of f). By the theorem, after restricting to an open neighborhood of the non-klt locus of (X, f^c) (this is the union of the centers of the divisors E_i computing $\text{lct}(f)$), we have $\text{lct}(g) = c$. In addition, the proof shows that every divisor over X that computes $\text{lct}(g)$ also computes $\text{lct}(f)$.

Theorem 3.1 can easily be extended to ideals, as stated in Theorem 1.4, as follows.

Proof of Theorem 1.4. We may assume that X is affine. Again, it is immediate to see that the hypothesis implies that $\text{lct}(\mathfrak{b}) \leq \text{lct}(\mathfrak{a})$. In order to prove the reverse inequality, let N be an integer larger than $\text{lct}(\mathfrak{a})$, and choose N general linear combinations f_1, \dots, f_N of a fixed set of generators of \mathfrak{a} . Note in particular that $\text{ord}_E(f_i) = \text{ord}_E(\mathfrak{a})$ for all i . Moreover, if $f := f_1 \dots f_N$, then $\text{lct}(f) = \text{lct}(\mathfrak{a})/N$ and E computes $\text{lct}(f)$ (see, for example, [Laz, Proposition 9.2.26]).

By assumption, we can write $f_i = g_i + h_i$, with $g_i \in \mathfrak{b}$ and $h_i \in \mathfrak{q}$. Note that we have $\text{ord}_E(h_i) > \text{ord}_E(\mathfrak{a})$, and hence $\text{ord}_E(g_i) = \text{ord}_E(\mathfrak{a})$, for every i . If $g := g_1 \dots g_N$, then we can write

$$f - g = h_1 f_2 \dots f_N + g_1 h_2 f_3 \dots f_N + \dots + g_1 g_2 \dots g_{N-1} h_N.$$

Since all terms in the above sum have order along E larger than $\text{ord}_E(f)$, we conclude by Theorem 3.1 that after possibly replacing X by an open neighborhood of the center of E , we have $\text{lct}(g) \geq \text{lct}(f)$. Since $g \in \mathfrak{b}^N$, it follows that $\text{lct}(\mathfrak{b}) \geq \text{lct}(\mathfrak{a})$. \square

Corollary 3.5. *Let $X = \text{Spec}(R)$, where $R = k[[x_1, \dots, x_n]]$, and let \mathfrak{a} and \mathfrak{b} proper ideals in R . Suppose that E is a divisor over X with center equal to the closed point, such that*

E computes $\text{lct}(\mathfrak{a})$. If $\mathfrak{b} + \mathfrak{q} = \mathfrak{a} + \mathfrak{q}$, where $\mathfrak{q} = \{h \in R \mid \text{ord}_E(h) > \text{ord}_E(\mathfrak{a})\}$, then $\text{lct}(\mathfrak{b}) = \text{lct}(\mathfrak{a})$.

Proof. It is enough to show that $\text{lct}(\mathfrak{b} + \mathfrak{m}^N) = \text{lct}(\mathfrak{a} + \mathfrak{m}^N)$ for all $N \gg 0$, where \mathfrak{m} denotes the maximal ideal in R (we use the fact that $\text{lct}(\mathfrak{b}) = \lim_{N \rightarrow \infty} \text{lct}(\mathfrak{b} + \mathfrak{m}^N)$ and $\text{lct}(\mathfrak{a}) = \lim_{N \rightarrow \infty} \text{lct}(\mathfrak{a} + \mathfrak{m}^N)$, see [dFM, Proposition 2.5]). Since the center of E is equal to the closed point, there is a divisor F over \mathbf{A}^n with center the origin such that E is obtained from F by base-change with respect to $\text{Spec}(R) \rightarrow \mathbf{A}^n$. If $\tilde{\mathfrak{a}}_N := (\mathfrak{a} + \mathfrak{m}^N) \cap k[x_1, \dots, x_n]$ and $\tilde{\mathfrak{b}}_N := (\mathfrak{b} + \mathfrak{m}^N) \cap k[x_1, \dots, x_n]$, then $\mathfrak{a} + \mathfrak{m}^N = \tilde{\mathfrak{a}}_N \cdot R$ and $\mathfrak{b} + \mathfrak{m}^N = \tilde{\mathfrak{b}}_N \cdot R$. Hence $\text{lct}(\mathfrak{a} + \mathfrak{m}^N) = \text{lct}_0(\tilde{\mathfrak{a}}_N)$ and $\text{lct}(\mathfrak{b} + \mathfrak{m}^N) = \text{lct}_0(\tilde{\mathfrak{b}}_N)$ (see, for example, [dFM, Corollary 2.8]).

On the other hand, we have $\text{lct}(\mathfrak{a} + \mathfrak{m}^N) \geq \text{lct}(\mathfrak{a})$ for every N , and $\text{lct}(\mathfrak{a} + \mathfrak{m}^N) \leq \text{lct}(\mathfrak{a})$ for $N > \text{ord}_E(\mathfrak{a})$. It follows that for such N we have $\text{lct}(\mathfrak{a} + \mathfrak{m}^N) = \text{lct}(\mathfrak{a})$, and furthermore, E computes $\text{lct}(\mathfrak{a} + \mathfrak{m}^N)$. Therefore F computes $\text{lct}_0(\tilde{\mathfrak{a}}_N)$. If $N > \text{ord}_E(\mathfrak{a})$, then $\text{ord}_F(\tilde{\mathfrak{a}}_N) = \text{ord}_E(\mathfrak{a})$, and

$$(x_1, \dots, x_n)^N \subseteq \tilde{\mathfrak{q}} := \{h \in k[x_1, \dots, x_n] \mid \text{ord}_F(h) > \text{ord}_F(\tilde{\mathfrak{a}}_N)\} = \mathfrak{q} \cap k[x_1, \dots, x_n].$$

We deduce that $\tilde{\mathfrak{b}}_N + \tilde{\mathfrak{q}} = \tilde{\mathfrak{a}}_N + \tilde{\mathfrak{q}}$, hence by Theorem 1.4 we have $\text{lct}_0(\tilde{\mathfrak{b}}_N) = \text{lct}_0(\tilde{\mathfrak{a}}_N)$. We conclude that $\text{lct}(\mathfrak{b} + \mathfrak{m}^N) = \text{lct}(\mathfrak{a} + \mathfrak{m}^N)$ for all $N \gg 0$, and therefore $\text{lct}(\mathfrak{b}) = \text{lct}(\mathfrak{a})$. \square

4. GENERIC LIMITS OF SEQUENCES OF IDEALS

In this section we review the construction from [Kol2], extending it from sequences of power series to sequences of ideals. In fact, we will need a version dealing with several such sequences. The goal is to associate to these sequences of ideals in a fixed polynomial ring or ring of power series, corresponding “limit” ideals through a sequence of \mathfrak{m} -adic approximations and field extensions.

For the sake of notation we only treat the case of two sequences. This is the only case needed in the paper. It will be however clear that the construction can be carried out for any given number of sequences. We also note that by taking the second sequence to be constant, the construction given below reduces in particular to a construction of generic limits for just one sequence. Furthermore, the assertions in Proposition 4.4 and Corollary 4.5 below reduce to statements about one sequence by taking $q = 0$.

Let $R = k[[x_1, \dots, x_n]]$ be the ring of formal power series in n variables with coefficients in an algebraically closed field k , and let \mathfrak{m} be its maximal ideal. If $k \subset K$ is a field extension, then we put $R_K := K[[x_1, \dots, x_n]]$ and $\mathfrak{m}_K := \mathfrak{m} \cdot R_K$.

For every $d \geq 1$, we consider the quotient homomorphism $R \rightarrow R/\mathfrak{m}^d$. We identify the ideals in R/\mathfrak{m}^d with the ideals in R containing \mathfrak{m}^d . Let \mathcal{H}_d be the Hilbert scheme parametrizing the ideals in R/\mathfrak{m}^d , with the reduced scheme structure. Since $\dim_k(R/\mathfrak{m}^d) < \infty$, \mathcal{H}_d is an algebraic variety. Note that for every field extension K of k , the K -valued

points of $\mathcal{H}_d \times \mathcal{H}_d$ correspond to pairs of ideals in R_K containing \mathfrak{m}_K^d . Mapping a pair of ideals in R/\mathfrak{m}^d to the pair consisting of their images in R/\mathfrak{m}^{d-1} gives a surjective map $t_d: \mathcal{H}_d \times \mathcal{H}_d \rightarrow \mathcal{H}_{d-1} \times \mathcal{H}_{d-1}$. This is not a morphism. However, by Generic Flatness we can cover $\mathcal{H}_d \times \mathcal{H}_d$ by disjoint locally closed subsets such that the restriction of t_d to each of these subsets is a morphism. In particular, for every irreducible closed subset $Z \subseteq \mathcal{H}_d \times \mathcal{H}_d$, the map t_d induces a rational map $Z \dashrightarrow \mathcal{H}_{d-1} \times \mathcal{H}_{d-1}$.

Suppose now that $(\mathfrak{a}_i)_{i \in I_0}$ and $(\mathfrak{b}_i)_{i \in I_0}$ are sequences of ideals in R indexed by the set $I_0 = \mathbf{Z}_+$. We consider sequences of irreducible closed subsets $Z_d \subseteq \mathcal{H}_d \times \mathcal{H}_d$ for $d \geq 1$ such that

- (\star) For every $d \geq 1$, the projection t_{d+1} induces a dominant rational map $\varphi_{d+1}: Z_{d+1} \dashrightarrow Z_d$.
- ($\star\star$) For every $d \geq 1$, there are infinitely many i with $(\mathfrak{a}_i + \mathfrak{m}^d, \mathfrak{b}_i + \mathfrak{m}^d) \in Z_d$, and the set of such $(\mathfrak{a}_i + \mathfrak{m}^d, \mathfrak{b}_i + \mathfrak{m}^d)$ is dense in Z_d .

Given such a sequence $(Z_d)_{d \geq 1}$, we define inductively nonempty open subsets $Z_d^\circ \subseteq Z_d$, and a nested sequence of infinite subsets

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots,$$

as follows. We put $Z_1^\circ = Z_1$ and $I_1 = \{i \in I_0 \mid (\mathfrak{a}_i + \mathfrak{m}, \mathfrak{b}_i + \mathfrak{m}) \in Z_1^\circ\}$. For $d \geq 2$, let $Z_d^\circ = \varphi_d^{-1}(Z_{d-1}^\circ) \subseteq \text{Domain}(\varphi_d)$ and $I_d = \{i \in I_0 \mid (\mathfrak{a}_i + \mathfrak{m}^d, \mathfrak{b}_i + \mathfrak{m}^d) \in Z_d^\circ\}$. It follows by induction on d that Z_d° is open in Z_d , and condition ($\star\star$) implies that each I_d is infinite. Furthermore, it is clear that $I_d \supseteq I_{d+1}$.

Sequences $(Z_d)_{d \geq 1}$ satisfying (\star) and ($\star\star$) can be constructed as follows. We first choose a minimal irreducible closed subset $Z_1 \subseteq \mathcal{H}_1 \times \mathcal{H}_1$ with the property that it contains $(\mathfrak{a}_i + \mathfrak{m}, \mathfrak{b}_i + \mathfrak{m})$ for infinitely many indices $i \in I_0$. We set $J_1 = \{i \in I_0 \mid (\mathfrak{a}_i + \mathfrak{m}, \mathfrak{b}_i + \mathfrak{m}) \in Z_1\}$. By construction, J_1 is an infinite set and Z_1 is the closure of $\{(\mathfrak{a}_i + \mathfrak{m}, \mathfrak{b}_i + \mathfrak{m}) \mid i \in J_1\}$. Next, we choose a minimal closed subset $Z_2 \subseteq \mathcal{H}_2 \times \mathcal{H}_2$ that contains $(\mathfrak{a}_i + \mathfrak{m}^2, \mathfrak{b}_i + \mathfrak{m}^2)$ for infinitely many i in J_1 (note that by minimality, Z_2 is irreducible). By construction, the set $J_2 = \{i \in J_1 \mid (\mathfrak{a}_i + \mathfrak{m}^2, \mathfrak{b}_i + \mathfrak{m}^2) \in Z_2\}$ is infinite, and Z_2 is the closure of $\{(\mathfrak{a}_i + \mathfrak{m}^2, \mathfrak{b}_i + \mathfrak{m}^2) \mid i \in J_2\}$. As we have seen, t_2 induces a rational map $\varphi_2: Z_2 \dashrightarrow Z_1$. Note that by the minimality in the choice of Z_1 , the rational map φ_2 is dominant. Repeating this process we select a sequence $(Z_d)_{d \geq 1}$ that satisfies (\star) and ($\star\star$) above.

Suppose now that we have a sequence $(Z_d)_{d \geq 1}$ with these two properties. The rational maps φ_d induce a nested sequence of function fields $k(Z_d)$. Let $K := \bigcup_{d \geq 1} k(Z_d)$. Each morphism $\text{Spec}(K) \rightarrow Z_d \subseteq \mathcal{H}_d \times \mathcal{H}_d$ corresponds to a pair of ideals \mathfrak{a}'_d and \mathfrak{b}'_d in R_K containing \mathfrak{m}_K^d , and the compatibility between these morphisms implies that there are (unique) ideals \mathfrak{a} and \mathfrak{b} in R_K such that $\mathfrak{a}'_d = \mathfrak{a} + \mathfrak{m}_K^d$ and $\mathfrak{b}'_d = \mathfrak{b} + \mathfrak{m}_K^d$ for all d .

Definition 4.1. With the above notation, we say that the pair of ideals $(\mathfrak{a}, \mathfrak{b})$ is a *generic limit* of the sequence of pairs of ideals $(\mathfrak{a}_i, \mathfrak{b}_i)_{i \geq 1}$.

Remark 4.2. The reader may compare the above construction with a similar one that can be used to show that every sequence $(x_i)_{i \geq 1}$, with all x_i in a closed bounded interval $L_0 = [a, b]$, contains a convergent subsequence. In that case, one also constructs by induction closed bounded intervals $L_d = [u_d, w_d]$ with $L_d \subseteq L_{d-1}$ and $(w_d - u_d) < \varepsilon_d$ (for some sequence ε_d converging to zero), and infinite subsets $I_d \subseteq I_{d-1} \subseteq I_0 = \mathbf{Z}_+$, such that $x_i \in L_d$ for all $i \in I_d$. With this notation, it is then clear that $(x_i)_{i \geq 1}$ contains a subsequence converging to $\sup_d u_d = \inf_d w_d$.

We list in the next lemma some easy properties of generic limits. The proof is straightforward, so we omit it.

Lemma 4.3. *Let $(\mathfrak{a}_i)_{i \geq 1}$ and $(\mathfrak{b}_i)_{i \geq 1}$ be sequences of ideals in R , and let $(\mathfrak{a}, \mathfrak{b})$ be a generic limit as above, with $\mathfrak{a}, \mathfrak{b} \subseteq R_K$.*

- i) *If $\mathfrak{b}_i = \mathfrak{c}$ for every i , where $\mathfrak{c} \subseteq R$ is a fixed ideal, then $\mathfrak{b} = \mathfrak{c} \cdot R_K$.*
- ii) *If $q \geq 1$ is such that $\mathfrak{a}_i \subseteq \mathfrak{m}^q$ for every i , then $\mathfrak{a} \subseteq \mathfrak{m}_K^q$.*
- iii) *If $q \geq 1$ is such that $\mathfrak{a}_i \not\subseteq \mathfrak{m}^q$ for every i , then $\mathfrak{a} \not\subseteq \mathfrak{m}_K^q$.*
- iv) *If $\mathfrak{a} = (0)$, then for every $q \geq 1$ there are infinitely many d such that $\mathfrak{a}_d \subseteq \mathfrak{m}^q$.*

In the following proposition we keep the notation used in the definition of generic limits. Recall that we have also defined the nested sequence of infinite sets $(I_d)_{d \geq 1}$.

Proposition 4.4. *Let $\mathfrak{a}, \mathfrak{b} \subseteq R_K$ be such that $(\mathfrak{a}, \mathfrak{b})$ is a generic limit of the sequence $(\mathfrak{a}_i, \mathfrak{b}_i)_{i \geq 1}$ of pairs of ideals in R . Assume that $\mathfrak{a}_i, \mathfrak{b}_i \neq R$ for all i . For every d there is an infinite subset $I_d^\circ \subseteq I_d$ such that for all nonnegative integers p and q*

$$\text{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b} + \mathfrak{m}_K^d)^q) = \text{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^p \cdot (\mathfrak{b}_i + \mathfrak{m}^d)^q) \quad \text{for every } i \in I_d^\circ.$$

Moreover, if E is a divisor over $\text{Spec}(R_K)$, with center at the closed point and computing $\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q)$ for some nonnegative integers p and q , then there is an integer d_E such that for every $d \geq d_E$ the following holds: there is an infinite subset $I_d^E \subseteq I_d^\circ$, and for every $i \in I_d^E$ a divisor E_i over $\text{Spec}(R)$ computing $\text{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^p \cdot (\mathfrak{b}_i + \mathfrak{m}^d)^q)$, such that $\text{ord}_E(\mathfrak{a} + \mathfrak{m}_K^d) = \text{ord}_{E_i}(\mathfrak{a}_i + \mathfrak{m}^d)$ and $\text{ord}_E(\mathfrak{b} + \mathfrak{m}_K^d) = \text{ord}_{E_i}(\mathfrak{b}_i + \mathfrak{m}_K^d)$.

In the second assertion in the proposition, both d_E and the sets I_d^E also depend on p and q , while E_i also depends on d .

Proof. Note that every ideal of the form $\mathfrak{c} + \mathfrak{m}^d$ can be considered as the ideal of a scheme on \mathbf{A}_k^n supported at the origin, and the log canonical threshold computed in $\text{Spec}(R)$ is the same as when computed in \mathbf{A}_k^n (cf. [dFM, Corollary 2.8]). Of course, the same holds if we replace k by K . Whenever we can, we adopt this alternative point of view, since base change works better in this setting (by base change an affine space becomes another affine space).

On $\mathcal{H}_d \times \mathbf{A}_k^n$ we have the universal family of ideals \mathcal{I} . Pulling this back via the two projections $\mathcal{H}_d \times \mathcal{H}_d \times \mathbf{A}_k^n \rightarrow \mathcal{H}_d \times \mathbf{A}_k^n$, and then restricting to $Z_d \times \mathbf{A}_k^n$ gives the ideals \mathcal{I}_d

and \mathcal{J}_d on $Z_d \times \mathbf{A}_k^n$. Let $\mu_d: Y_d \rightarrow Z_d \times \mathbf{A}_k^n$ be a log resolution of the product $\mathcal{I}_d \cdot \mathcal{J}_d$, and let \mathcal{E} be the relevant simple normal crossings divisor on Y_d . By Generic Smoothness, there is a nonempty open subset $U_d \subseteq Z_d$ such that the induced map $Y_d \rightarrow Z_d$ is smooth over U_d , and furthermore, \mathcal{E} has relative simple normal crossings over U_d . In this case, the fiber of $Y_d \rightarrow Z_d$ over a point in U_d corresponding to a pair of ideals $(\mathfrak{c}_1, \mathfrak{c}_2)$ gives a log resolution of the ideal $\mathfrak{c}_1 \cdot \mathfrak{c}_2$ in \mathbf{A}_k^n . It follows that for every p and q , the log canonical threshold $\text{lct}(\mathfrak{c}_1^p \cdot \mathfrak{c}_2^q)$ is independent of the point $(\mathfrak{c}_1, \mathfrak{c}_2) \in U_d$. Moreover, it is equal to the similar log canonical threshold computed for the pair of ideals parametrized by the generic point of Z_d . These are ideals in $k(Z_d)[x_1, \dots, x_n]$ whose extensions to $K[x_1, \dots, x_n]$ are $\mathfrak{a} + \mathfrak{m}_K^d$ and $\mathfrak{b} + \mathfrak{m}_K^d$. We thus take $I_d^\circ \subset I_d$ to consist of those i for which $(\mathfrak{a}_i + \mathfrak{m}_K^d, \mathfrak{b}_i + \mathfrak{m}_K^d)$ is in U_d . Condition $(\star\star)$ on the sequence $(Z_d)_{d \geq 1}$ implies that I_d° is an infinite set.

For the second assertion in the proposition, observe first that since E has center equal to the closed point, there is a divisor F over \mathbf{A}_K^n with center at the origin, such that E is obtained from F by base-change with respect to $\text{Spec}(R_K) \rightarrow \mathbf{A}_K^n$. Given an ideal $\mathfrak{c} + \mathfrak{m}_K^d \subset R_K$, the divisor E computes the log canonical threshold of this ideal if and only if F computes the log canonical threshold of the corresponding ideal in $K[x_1, \dots, x_n]$.

Note that the divisor F , a priori defined over K , is in fact defined over a subextension L of K/k , of finite type over k . Let $d_E > \text{ord}_E(\mathfrak{a}^p \cdot \mathfrak{b}^q)$ be an integer such that F is defined over $k(Z_{d_E})$. For $d \geq d_E$, we have $\text{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b} + \mathfrak{m}_K^d)^q) = \text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q)$, and E computes both these log canonical thresholds: for this one argues as in the beginning of the proof of Theorem 3.1, observing that in this case we have $\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q) \leq \text{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b} + \mathfrak{m}_K^d)^q)$ due to the inclusion $\mathfrak{a}^p \cdot \mathfrak{b}^q \subseteq (\mathfrak{a} + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b} + \mathfrak{m}_K^d)^q$.

On the other hand, for every such d we can find a nonempty open subset $W_d \subseteq Z_d$ and a log resolution $\nu_d: Y'_d \rightarrow W_d \times \mathbf{A}_k^n$ of the restriction of $\mathcal{I}_d \cdot \mathcal{J}_d$ to $W_d \times \mathbf{A}_k^n$, such that F is obtained from a divisor \mathcal{F}' on Y'_d by base-change with respect to the composition

$$\mathbf{A}_K^n \rightarrow \mathbf{A}_{k(Z_d)}^n \rightarrow W_d \times \mathbf{A}_k^n.$$

Arguing as in the first part of the proof, we see that after possibly replacing W_d by a smaller open subset, we may assume that Y'_d is smooth over W_d , and furthermore, that the relevant divisor \mathcal{E}' has relative simple normal crossings over W_d . Note that \mathcal{F}' is a component of \mathcal{E}' .

Let $I_d^E := \{i \in I_d^\circ \mid (\mathfrak{a}_i + \mathfrak{m}_K^d, \mathfrak{b}_i + \mathfrak{m}_K^d) \in W_d\}$. Again, condition $(\star\star)$ on the sequence $(Z_d)_{d \geq 1}$ implies that I_d^E is infinite. Since F computes the log canonical threshold of the (extension to $K[x_1, \dots, x_n]$ of the) suitable product corresponding to the pair of ideals parametrized by the generic point of W_d , it follows that if $i \in I_d^E$, and F_i is a connected component of the fiber of \mathcal{F}' over the point in W_d representing $(\mathfrak{a}_i + \mathfrak{m}_K^d, \mathfrak{b}_i + \mathfrak{m}_K^d)$, then F_i computes $\text{lct}((\mathfrak{a}_i + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b}_i + \mathfrak{m}_K^d)^q)$. Moreover, we have $\text{ord}_F(\mathfrak{a} + \mathfrak{m}_K^d) = \text{ord}_{F_i}(\mathfrak{a}_i + \mathfrak{m}_K^d)$ and $\text{ord}_F(\mathfrak{b} + \mathfrak{m}_K^d) = \text{ord}_{F_i}(\mathfrak{b}_i + \mathfrak{m}_K^d)$. If E_i is obtained from F_i by base-change via $\text{Spec}(R) \rightarrow \mathbf{A}_K^n$, then E_i satisfies the requirement in the proposition. \square

Corollary 4.5. *With the above notation, for every sequence $(i_d)_{d \geq 1}$ with $i_d \in I_d^\circ$, we have $\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q) = \lim_{d \rightarrow \infty} \text{lct}(\mathfrak{a}_{i_d}^p \cdot \mathfrak{b}_{i_d}^q)$ for all nonnegative integers p and q . In particular, if the sequence $(\text{lct}(\mathfrak{a}_{i_d}^p \cdot \mathfrak{b}_{i_d}^q))_{d \geq 1}$ is convergent, then it converges to $\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q)$.*

Proof. Recall the following basic fact: if \mathfrak{c} and \mathfrak{c}' are proper ideals in R , with $\mathfrak{c} + \mathfrak{m}^d = \mathfrak{c}' + \mathfrak{m}^d$, then

$$|\text{lct}(\mathfrak{c}) - \text{lct}(\mathfrak{c}')| \leq \frac{n}{d}$$

(see [dFM, Corollary 2.10]). Note that this equality also holds when \mathfrak{c} or \mathfrak{c}' are zero. Of course, a similar result holds for ideals in R_K . It follows from Proposition 4.4 that for every $d \geq 1$ we have

$$\begin{aligned} |\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q) - \text{lct}(\mathfrak{a}_{i_d}^p \cdot \mathfrak{b}_{i_d}^q)| &\leq |\text{lct}(\mathfrak{a}^p \cdot \mathfrak{b}^q) - \text{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^p \cdot (\mathfrak{b} + \mathfrak{m}_K^d)^q)| \\ &\quad + |\text{lct}((\mathfrak{a}_{i_d} + \mathfrak{m}^d)^p \cdot (\mathfrak{b}_{i_d} + \mathfrak{m}^d)^q) - \text{lct}(\mathfrak{a}_{i_d}^p \cdot \mathfrak{b}_{i_d}^q)| \leq \frac{2n}{d}. \end{aligned}$$

The assertion in the proposition is an immediate consequence. \square

Remark 4.6. It is clear that both the construction and the above results generalize in an obvious way to any finite number of sequences of ideals.

5. LOG CANONICAL THRESHOLDS ON SMOOTH VARIETIES

This section is devoted to the proof of Theorem 1.1. For completeness, we also include the proof of the smooth case of Kollár's Accumulation Conjecture [Kol1], which is already known by the results in [dFM, Kol2]: the case of limits of decreasing sequences was first treated in [dFM], and the proof was completed in [Kol2] where the case of (potential) limits of increasing sequences was also treated.

Theorem 5.1. *For every n , the set $\mathcal{T}_n^{\text{sm}}$ satisfies the ascending chain condition, and its set of accumulation points is $\mathcal{T}_{n-1}^{\text{sm}}$.*

We start with an easy lemma that can be used to replace an ideal by another ideal with the same log canonical threshold, and such that this log canonical threshold is computed by a divisor having a zero-dimensional center.

Lemma 5.2. *Let \mathfrak{a} be an ideal contained in the maximal ideal \mathfrak{m}_K of $K[[x_1, \dots, x_n]]$. We put $q := \max\{t \geq 0 \mid \text{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^t) = \text{lct}(\mathfrak{a})\}$.*

- i) *We have $q \in \mathbf{Q}_{\geq 0}$.*
- ii) *If we write $q = r/s$, for nonnegative integers r and s , then $\text{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r) = \frac{\text{lct}(\mathfrak{a})}{s}$, and this log canonical threshold is computed by a divisor with center equal to the closed point.*
- iii) *We have $q = 0$ if and only if $\text{lct}(\mathfrak{a})$ is computed by a divisor with center over the closed point.*

Proof. Let $\pi: Y \rightarrow X = \operatorname{Spec}(K[[x_1, \dots, x_n]])$ be a log resolution of $\mathfrak{a} \cdot \mathfrak{m}_K$, and write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}(-\sum_i a_i E_i)$, $\mathfrak{m}_K \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_i b_i E_i)$, and $K_{Y/X} = \mathcal{O}_Y(-\sum_i k_i E_i)$. Let I denote the set of those i for which E_i has center equal to the closed point, that is, such that $b_i > 0$.

Let $c = \operatorname{lct}(\mathfrak{a})$. Note that we have $\operatorname{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^t) \leq c$ for every $t \geq 0$. Furthermore, $\operatorname{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^t) \geq c$ if and only if

$$k_i + 1 \geq c(a_i + tb_i)$$

for all i . If $i \notin I$, then $b_i = 0$ and this inequality holds for all t . We conclude that

$$q = \min \left\{ \frac{k_i + 1 - ca_i}{cb_i} \mid i \in I \right\}.$$

This shows that $q \in \mathbf{Q}$. Moreover, if $i \in I$ is such that this minimum is achieved, then E_i computes $\operatorname{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r)$, and E_i has center equal to the closed point. The assertion in iii) is clear. \square

Proof of Theorem 5.1. Let $(c_i)_{i \geq 1}$ be a strictly monotone sequence with terms in $\mathcal{T}_n^{\operatorname{sm}}$, and let $c = \lim_{i \rightarrow \infty} c_i$ (the limit is finite, since $\mathcal{T}_n^{\operatorname{sm}}$ is bounded above by n). For every i we can select an ideal $\tilde{\mathfrak{a}}_i \subseteq (x_1, \dots, x_n) \subset k[[x_1, \dots, x_n]]$ with $\operatorname{lct}_0(\tilde{\mathfrak{a}}_i) = c_i$. Let $\mathfrak{a}_i = \tilde{\mathfrak{a}}_i \cdot k[[x_1, \dots, x_n]]$. Consider a generic limit $(\mathfrak{a}, \mathfrak{b})$ of the sequence of pairs of ideals $(\mathfrak{a}_i, \mathfrak{m})_{i \geq 1}$, constructed as in Section 4, with $\mathfrak{a}, \mathfrak{b} \subseteq K[[x_1, \dots, x_n]]$. Note that by Lemma 4.3, we have $\mathfrak{a} \subseteq \mathfrak{m}_K$, and $\mathfrak{b} = \mathfrak{m}_K$. Since $\operatorname{lct}(\mathfrak{a}_i) = \operatorname{lct}_0(\tilde{\mathfrak{a}}_i)$ (see, for example, [dFM, Proposition 2.9]), it follows from Corollary 4.5 that $\operatorname{lct}(\mathfrak{a}) = c$. If $c = 0$, then the sequence $(c_i)_{i \geq 1}$ can't be strictly increasing. Furthermore, we have $0 \in \mathcal{T}_{n-1}^{\operatorname{sm}}$ (note indeed that $n > 0$), hence this case is clear, and we may assume that $c > 0$. In particular, $\mathfrak{a} \neq (0)$.

Let q be the rational number attached to \mathfrak{a} as in the lemma, and write $q = r/s$, with r and s nonnegative integers. By construction, we have

$$\operatorname{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r) = \frac{1}{s} \operatorname{lct}(\mathfrak{a}).$$

On the other hand, we certainly have

$$\operatorname{lct}(\mathfrak{a}_i^s \cdot \mathfrak{m}^r) \leq \frac{1}{s} \operatorname{lct}(\mathfrak{a}_i) \quad \text{for every } i.$$

Note in particular that if $(c_i)_{i \geq 1}$ is a strictly increasing sequence, then $\operatorname{lct}(\mathfrak{a}_i^s \cdot \mathfrak{m}^r) < \operatorname{lct}(\mathfrak{a}^s \cdot \mathfrak{m}^r)$ for every i .

By the choice of q , $\operatorname{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r)$ is computed by a divisor E which lies over the closed point of $\operatorname{Spec}(K[[x_1, \dots, x_n]])$. Fix any $d \geq d_E$, with d_E associated to E, s, r and to the sequence $(\mathfrak{a}_i, \mathfrak{m})_{i \geq 1}$ by Proposition 4.4. As in the proof of that proposition, we may and will assume that $d_E > \operatorname{ord}_E(\mathfrak{a}^s \cdot \mathfrak{m}_K^r)$, so that for all $d \geq d_E$ we have $\operatorname{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r) = \operatorname{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^s \cdot \mathfrak{m}_K^r)$, and E computes both log canonical thresholds.

By Proposition 4.4, there is an infinite set $I_d^E \subseteq \mathbf{Z}_+$ such that for every $i \in I_d^E$ we have $\operatorname{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^s \cdot \mathfrak{m}_K^r) = \operatorname{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r)$, and moreover, there is a divisor E_i over

$\text{Spec}(k[[x_1, \dots, x_n]])$ computing $\text{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r)$, and such that

$$\text{ord}_{E_i}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r) = \text{ord}_E((\mathfrak{a} + \mathfrak{m}_K^d)^s \cdot \mathfrak{m}_K^r) = \text{ord}_E(\mathfrak{a}^s \cdot \mathfrak{m}_K^r).$$

Since E_i is a divisor computing $\text{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r)$, its center is equal to the closed point. Furthermore, by our condition on d we have

$$\text{ord}_{E_i}(\mathfrak{m}^d) \geq d > \text{ord}_E(\mathfrak{a}^s \cdot \mathfrak{m}^r) = \text{ord}_{E_i}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r),$$

hence Corollary 3.5 gives for every $i \in I_d^E$

$$\text{lct}(\mathfrak{a}_i^s \cdot \mathfrak{m}^r) = \text{lct}((\mathfrak{a}_i + \mathfrak{m}^d)^s \cdot \mathfrak{m}^r) = \text{lct}((\mathfrak{a} + \mathfrak{m}_K^d)^s \cdot \mathfrak{m}_K^r) = \text{lct}(\mathfrak{a}^s \cdot \mathfrak{m}_K^r).$$

It follows from the above discussion that $(c_i)_{i \geq 1}$ cannot be a strictly increasing sequence, which proves that $\mathcal{T}_n^{\text{sm}}$ satisfies the ascending chain condition. By exclusion, $(c_i)_{i \geq 1}$ has to be a strictly decreasing sequence. Since the sequence $(\text{lct}(\mathfrak{a}_i^s \cdot \mathfrak{m}^r))_{i \geq 1}$ has repeating terms, we deduce that $q > 0$. Equivalently, $\text{lct}(\mathfrak{a})$ is not computed by any divisor with center at the closed point. Therefore, if F is a divisor over $\text{Spec}(K[[x_1, \dots, x_n]])$ computing $\text{lct}(\mathfrak{a})$, then the center of F in $\text{Spec}(K[[x_1, \dots, x_n]])$ is positive dimensional, and hence, after localizing at its generic point, we see that $\text{lct}(\mathfrak{a}) \in \mathcal{T}_{n-1}^{\text{sm}}$ (cf. [dFM, Propositions 2.11 and 3.1]). As it is easy and well-known that, conversely, every element in $\mathcal{T}_{n-1}^{\text{sm}}$ is an accumulation point of $\mathcal{T}_n^{\text{sm}}$, we conclude that $\mathcal{T}_{n-1}^{\text{sm}}$ is equal to the set of accumulation points of $\mathcal{T}_n^{\text{sm}}$. \square

The following proposition allows us to reduce log canonical thresholds on varieties with quotient singularities to log canonical thresholds on smooth varieties. We say that a variety X has *quotient singularities* at $p \in X$ if there is a smooth variety U , a finite group G acting on U , and a point $q \in V = U/G$ such that the two completions $\widehat{\mathcal{O}_{X,p}}$ and $\widehat{\mathcal{O}_{V,q}}$ are isomorphic as k -algebras. We say that X has quotient singularities if it has quotient singularities at every point.

In the above definition, one can assume that U is an affine space and that the action of G is linear. Furthermore, one can assume that G acts with no fixed points in codimension one (otherwise, we may replace G by G/H and U by U/H , where H is generated by all pseudoreflections in G , and by Chevalley's theorem [Che], the quotient U/H is again an affine space). Using Artin's approximation results (see Corollary 2.6 in [Art]), it follows that there is an étale neighborhood of p that is also an étale neighborhood of q . In other words, there is a variety W , a point $r \in W$, and étale maps $\varphi: W \rightarrow X$ and $\psi: W \rightarrow V$, such that $p = \varphi(r)$ and $q = \psi(r)$. After replacing φ by the composition

$$W \times_V U \rightarrow W \xrightarrow{\varphi} X,$$

we may assume that in fact we have an étale map $U/G \rightarrow X$ containing p in its image, with U smooth, and such that G acts on U without fixed points in codimension one. This reinterpretation of the definition of quotient singularities seems to be well-known to experts, but we could not find an explicit reference in the literature.

Proposition 5.3. *Let X be a variety with quotient singularities, and let \mathfrak{a} be a proper nonzero ideal on X . For every p in the zero-locus $V(\mathfrak{a})$ of \mathfrak{a} , there is a smooth variety U , a nonzero ideal \mathfrak{b} on U , and a point q in $V(\mathfrak{b})$ such that $\text{lct}_p(X, \mathfrak{a}) = \text{lct}_q(U, \mathfrak{b})$.*

Proof. Let us choose an étale map $\varphi: U/G \rightarrow X$ with $p \in \text{Im}(\varphi)$, where U is a smooth variety, and G is a finite group acting on U without fixed points in codimension one. Let $\tilde{\varphi}: U \rightarrow X$ denote the composition of φ with the quotient map. Since G acts without fixed points in codimension one, $\tilde{\varphi}$ is étale in codimension one, hence $K_U = \tilde{\varphi}^*(K_X)$. It follows from Proposition 5.20 in [KM] that if $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_U$, then the pair (X, \mathfrak{a}^t) is log canonical if and only if the pair (U, \mathfrak{b}^t) is log canonical (actually the result in *loc. cit.* only covers the case when \mathfrak{a} is locally principal, but one can easily reduce to this case, by taking a suitable product of general linear combinations of the local generators of \mathfrak{a}). We conclude that there is a point $q \in V(\mathfrak{b})$ such that $\text{lct}_p(X, \mathfrak{a}) = \text{lct}_q(U, \mathfrak{b})$. \square

It follows that $\mathcal{T}_n^{\text{quot}} = \mathcal{T}_n^{\text{sm}}$ for every n , and therefore we deduce by Theorem 5.1 that Shokurov's ACC Conjecture and Kollár's Accumulation Conjecture hold for log canonical thresholds on varieties with quotient singularities.

Corollary 5.4. *For every n , the set $\mathcal{T}_n^{\text{quot}}$ satisfies the ascending chain condition and its set of accumulation points is equal to $\mathcal{T}_{n-1}^{\text{quot}}$.*

Remark 5.5. At least over the complex numbers, one usually says that X has quotient singularities at p if the germ of analytic space (X, x) is isomorphic to M/G , where M is a complex manifold, and G is a finite group acting on M . It is not hard to check that in this context this definition is equivalent with the one we gave above.

6. LOG CANONICAL THRESHOLDS ON L.C.I. VARIETIES

In this section we prove that the ACC Conjecture holds for log canonical thresholds (and mixed log canonical thresholds) on l.c.i. varieties, and prove Theorem 1.2. We start with the case of mixed log canonical thresholds on smooth varieties.

Theorem 6.1. *For every n , the set $\mathcal{M}_n^{\text{sm}}$ satisfies the ascending chain condition.*

Proof. Suppose that $\mathcal{M}_n^{\text{sm}}$ contains a strictly increasing sequence $(c_i)_{i \geq 1}$. Let $c = \lim_{i \rightarrow \infty} c_i$ (which is finite, since $\mathcal{M}_n^{\text{sm}}$ is bounded above by n). We can find ideals $\tilde{\mathfrak{a}}_i, \tilde{\mathfrak{b}}_i \subseteq k[x_1, \dots, x_n]$, with $\tilde{\mathfrak{a}}_i \subseteq (x_1, \dots, x_n)$ and $\text{lct}_0(\tilde{\mathfrak{b}}_i) \geq 1$, such that $c_i = \text{lct}_{(\mathbb{A}^n, \tilde{\mathfrak{b}}_i), 0}(\tilde{\mathfrak{a}}_i)$. If \mathfrak{a}_i and \mathfrak{b}_i are the ideals generated by $\tilde{\mathfrak{a}}_i$ and, respectively, $\tilde{\mathfrak{b}}_i$ in $k[[x_1, \dots, x_n]]$, then $c_i = \text{lct}_{\mathfrak{b}_i}(\mathfrak{a}_i)$ by Remark 2.3. Consider a generic limit $(\mathfrak{a}, \mathfrak{b})$ of the sequence $(\mathfrak{a}_i, \mathfrak{b}_i)_{i \geq 1}$, constructed as in Section 4, with $\mathfrak{a}, \mathfrak{b} \subseteq K[[x_1, \dots, x_n]]$. By Corollary 4.5, $\text{lct}(\mathfrak{b})$ is a limit point of the sequence $(\text{lct}(\mathfrak{b}_i))_{i \geq 1}$, hence $\text{lct}(\mathfrak{b}) \geq 1$. Therefore $c' := \text{lct}_{\mathfrak{b}}(\mathfrak{a})$ is well defined.

Consider first any positive integers p and q such that $p/q < c$. By assumption, we have $c_i > p/q$ for all $i \gg 1$. Let $X = \text{Spec}(k[[x_1, \dots, x_n]])$. The pair $(X, \mathfrak{b}_i \cdot \mathfrak{a}_i^{p/q})$ is log

canonical, hence $\text{lct}(\mathfrak{b}_i^q \cdot \mathfrak{a}_i^p) \geq 1/q$, for all $i \gg 1$. It follows from Corollary 4.5 that there is a sequence $(i_d)_{d \geq 1}$ in \mathbf{Z}_+ such that

$$\text{lct}(\mathfrak{b}^q \cdot \mathfrak{a}^p) = \lim_{d \rightarrow \infty} \text{lct}(\mathfrak{b}_{i_d}^q \cdot \mathfrak{a}_{i_d}^p).$$

This implies in particular that $\text{lct}(\mathfrak{b}^q \cdot \mathfrak{a}^p) \geq 1/q$, and therefore $c' \geq p/q$. As this holds for every $p/q < c$, we conclude that $c' \geq c$.

On the other hand, since $c' \in \mathbf{Q}$, we may write $c' = r/s$ for positive integers r and s . It follows from Remark 2.2 that $\text{lct}(\mathfrak{b} \cdot \mathfrak{a}^{r/s}) = 1$, and thus $\text{lct}(\mathfrak{b}^s \cdot \mathfrak{a}^r) = 1/s$. Applying again Corollary 4.5, we find a sequence $(j_d)_{d \geq 1}$ in \mathbf{Z}_+ such that

$$\text{lct}(\mathfrak{b}^s \cdot \mathfrak{a}^r) = \lim_{d \rightarrow \infty} \text{lct}(\mathfrak{b}_{j_d}^s \cdot \mathfrak{a}_{j_d}^r).$$

The fact that $\mathcal{T}_n^{\text{sm}}$ satisfies the ascending chain condition (cf. Theorem 5.1) implies that there are infinitely many d such that $\text{lct}(\mathfrak{b}_{j_d}^s \cdot \mathfrak{a}_{j_d}^r) \geq 1/s$, hence $\text{lct}_{\mathfrak{b}_{j_d}}(\mathfrak{a}_{j_d}) \geq r/s$. For any such d we have

$$c' \geq c > c_{j_d} \geq \frac{r}{s} = c',$$

which is a contradiction. \square

In order to extend the above result to the case of ambient varieties with l.c.i. singularities, we use the following application of Inversion of Adjunction. This is the key tool that allows us to replace mixed log canonical thresholds on locally complete intersection varieties with the similar type of invariants on ambient smooth varieties.

Proposition 6.2. *Let A be a smooth irreducible variety over k , and $X \subset A$ a closed subvariety of pure codimension e , that is normal and locally a complete intersection. Suppose that \mathfrak{b} and \mathfrak{a} are ideals on A , with $\mathfrak{a} \neq \mathcal{O}_A$, and such that X is not contained in the union of the zero-loci of \mathfrak{b} and \mathfrak{a} .*

- i) *The pair $(X, \mathfrak{b}|_X)$ is log canonical if and only if for some open neighborhood U of X , the pair $(U, \mathfrak{b} \cdot \mathfrak{p}^e|_U)$ is log canonical, where \mathfrak{p} is the ideal defining X in A .*
- ii) *If $(X, \mathfrak{b}|_X)$ is log canonical, and if X intersects the zero-locus of \mathfrak{a} , then for some open neighborhood V of X we have*

$$\text{lct}_{\mathfrak{b}|_X}(X, \mathfrak{a}|_X) = \text{lct}_{\mathfrak{b}|_V \cdot \mathfrak{p}^e|_V}(V, \mathfrak{a}|_V).$$

Proof. Both assertions follow from Inversion of Adjunction (see Corollary 3.2 in [EM1]), as this says that for every nonnegative q , the pair $(X, (\mathfrak{b} \cdot \mathfrak{a}^q)|_X)$ is log canonical if and only if the pair $(A, \mathfrak{b} \cdot \mathfrak{a}^q \cdot \mathfrak{p}^e)$ is log canonical in some neighborhood of X . \square

The next fact, which must be well-known to the experts, allows us to control the dimension of the ambient variety in the process of replacing a mixed log canonical threshold on an l.c.i. variety by one on a smooth variety. Given a closed point $x \in X$, we denote by $T_x X$ the Zariski tangent space of X at x .

Proposition 6.3. *Let X be a locally complete intersection variety. If X is log canonical, then $\dim_k T_x X \leq 2 \dim X$ for every $x \in X$.*

Proof. Fix $x \in X$, and let $N = \dim T_x X$. After possibly replacing X by an open neighborhood of x , we may assume that we have a closed embedding of X in a smooth irreducible variety A , of codimension e , with $\dim A = N$. If $X = A$, then $N = \dim X$ and we are done.

Suppose now that $e \geq 1$. Since X is locally a complete intersection, it follows from Inversion of Adjunction (see Corollary 3.2 in [EM1]) that the pair (A, \mathfrak{p}^e) is log canonical, where \mathfrak{p} is the ideal of X in A . In particular, if E is the exceptional divisor of the blow-up A' of A at x , and ord_E is the corresponding valuation, then we have

$$N = 1 + \text{ord}_E(K_{A'/A}) \geq e \cdot \text{ord}_E(\mathfrak{p}) \geq 2e = 2(N - \dim X).$$

This gives $N \leq 2 \dim X$. □

We are now ready to prove Theorem 1.2, and hence Corollary 1.3.

Proof of Theorem 1.2. By Theorem 6.1, we know that $\mathcal{M}_n^{\text{sm}}$ satisfies the ascending chain condition for every n . Then it is clear that in order to prove that $\mathcal{M}_n^{\text{l.c.i.}}$ also satisfies the ascending chain condition for every n , it suffices to show that

$$\mathcal{M}_n^{\text{l.c.i.}} \subseteq \mathcal{M}_{2n}^{\text{sm}}.$$

Suppose that (X, \mathfrak{b}) is log canonical, with X locally a complete intersection of dimension n , and let $c = \text{lct}_{\mathfrak{b}}(\mathfrak{a})$. Let $x \in X$ be any point in the center of a divisor computing $\text{lct}_{\mathfrak{b}}(\mathfrak{a})$. For every open neighborhood U of x we have $\text{lct}_{\mathfrak{b}|_U}(U, \mathfrak{a}|_U) = c$. Since X is log canonical, it follows from Proposition 6.3 that $\dim_k T_x X \leq 2n$. After replacing X by a suitable neighborhood of x , we may assume that there is a closed embedding $X \hookrightarrow A$, where A is a smooth variety of dimension $2n$. Proposition 6.2 implies that after possibly replacing A by a neighborhood of X , we have $c = \text{lct}_{\mathfrak{b}_1 \cdot \mathfrak{p}^e}(\mathfrak{a}_1)$, where \mathfrak{p} is the ideal defining X in A , e is the codimension of X in A , and \mathfrak{b}_1 and \mathfrak{a}_1 are ideals in A whose restrictions to X give, respectively, \mathfrak{b} and \mathfrak{a} . Thus $c \in \mathcal{M}_{2n}^{\text{sm}}$. □

Proof of Corollary 1.3. It follows by Theorem 1.2, since $\mathcal{T}_n^{\text{l.c.i.}} \subseteq \mathcal{M}_n^{\text{l.c.i.}}$. □

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